

CHAPTER 2

A long time ago in a galaxy far far away.....

On the desert planet of TATOOINE,
A lone bounty hunter makes his way,
kicking up plumes of dust as he speeds
along the dunes.

Having negotiated with the JAWAS,
You have successfully helped return the parts
and helped repair your ship.

You now head to complete your quest
and deal with the desert pirates.
What dangers lie ahead for our mathematical hero....

NOTE: The submission format for all problems is a written answer with sufficient Justification.

Problem 1

As you speed along your sensors alert you that a hidden trap is ahead of ahead. Scanning ahead you notice that the trap is a deep trench in the shape of a convex pentagon with equal sides. Let the convex pentagon be TESLA with you positioned at vertex A. Given are $\angle SET = 136^\circ$, and $\angle ESL = 104^\circ$. Find $\angle TAL^\circ$ to safely evade the trap. **(400 Points)**

The pirate castle looms ahead hidden in between the dunes and slowly becomes bigger as you approach it. Surprisingly, none of the pirates attach you as you make your way into the castle. "Welcome, welcome to my abode!" Says a booming voice as a spotlight reveals the huge pirate king sitting on the throne. "I know what you're here for, in fact many hunters were here for the same reason as you - to stop me." You nod your head."I have an offer for you that will not end in any blood spilling. You solve four of my hardest questions and I will promise to not attack Mos Espa from here on. Fail to do so, and you shall be food to the Mudhorn, which you have no hope of escaping." Solve the four questions below as posed by the pirate king.

Problem 2

$$\text{Let } x = \frac{1}{1729} + \frac{1}{1730} + \cdots + \frac{1}{2024}.$$

Find the highest power of 19 that is present in $(\lfloor 10000x \rfloor)!$. **(400 Points)**

Problem 3

Consider the sequence of real numbers $a_1, a_2, \dots, a_{2022}$ such that $5a_{i+1} - a_i = 1 + 3a_i a_{i+1}$ for each $1 \leq i \leq 2022$ with $a_1 = a_{2023}$.

Find the sum of all possible values of $a_1 a_2 \dots a_{2022}$. **(300 Points)**

Problem 4

Consider the following congruent recurrences:

$$a_n \equiv a_{n-1} + 3n^2 \pmod{5} \text{ with } a_0 \equiv 4 \pmod{5}$$

$$a_n \equiv 6a_{n-2} - a_{n-1} \pmod{11} \text{ with } a_0 \equiv -1, a_1 \equiv 8 \pmod{11}$$

$$a_n \equiv 4a_{n-1} - 3a_{n-2} - 2 \pmod{7} \text{ with } a_0 \equiv 2, a_1 \equiv 5 \pmod{7}$$

What is a_{200} reduced modulo 385? **(400 Points)**

Problem 5

Let f be an expression defined on a sequence of positive integers x_1, x_2, \dots, x_k as follows :

$$f(x_1, x_2, \dots, x_k) = \frac{x_1^{x_2} x_2^{x_3} \dots x_{k-1}^{x_k}}{x_1! x_2! \dots x_k!}$$

Let S denote the set of positive integer sequences (with at least two terms) whose terms sum to 2023. If the sum of the values of f over all sequences in S is N .

Determine the last three digits of $((2024!)^N)^{2023^{2022^{2021 \dots^{2^1}}}}$. **(500 Points)**

Problem 6

"Wow! I truly am amazed, hunter. I promise to stop attacking the city. However I wish to play a game with you. I will maintain a set on $n+1$ binary numbers v_1, v_2, \dots, v_{n+1} of n digits each. You are tasked with producing an n -digit binary number which differs from each of my numbers. You are allowed to ask questions of the following form before you make your guess: "What is the j th digit of v_i ?" and I shall show you the digit. However you should try to ask as less questions as possible." What is the minimum number of questions you would need to ask to be guaranteed to win the game? **(300 Points)**

Problem 7

Having successfully stopped the pirate king you wish to indulge with the bounty award that you won by indulging in your favorite treats: cakes and cold drinks. The Cakes are available in three different price categories: \$199², \$ 221² and \$241². Whereas cold drinks are available at an affordable price of just \$1. Being a cake lover, you want to maximize your spending on cakes. (you will spend the whole amount on cakes if possible).

Let \$x be the maximum amount of money you should have to ensure that you can buy cakes and still have enough remaining to purchase cold drinks. Compute the last 3 digits of x^2 . **(300 Points)**