CHAPTER I

A long time ago in a galaxy far far away.....

The EVIL GALACTIC EMPIRE has fallen
and a NEW REPUBLIC has risen to take its place.
However, sinister agents are already at work
to undermine the fragile peace.

In the aftermath of the Galactic Empire's collapse,
the Outer Rim is a lawless expanse where the strong
thrive and the weak perish. A renowned bounty hunter,
you, makes his/her way across the tattered galaxy.

After youR encounter with a secret IMPERIAL SHIP,
having barely survived with STORMTROOPERS on your tail
YOU find yourself needing heavy repairs for your ship
and heading towards the desert planet of TATOOINE....

NOTE: The submission format for all problems is a written answer furnished with all the sufficient points and justifications.

Problem 1

Entering the atmosphere of Tatooine with heavy damage spells big trouble for you. As you fall at dizzying speeds, a warning light pops up, indicating that the ship is heating up due to the broken heat shields. you pull up the heat shield matrix, which is an $n * n$ matrix, denoted by A_{n*n}.
Let A_{n*n} be such that,

$$A_{i,j} = \frac{1}{min(i,j)}, 1 \leq i, j \leq n$$

To have any hope of fixing the heat shields in time before you burn to a crisp, you need to find two things:

a) Find $det(A)$ **(150 Points)**
b) Find trace of A^{-1} **(150 Points)**

Problem 2

You were able to fix the heat shields in time, and after a crash landing you struggle to make it out. The door is stuck shut and requires you to input the

password. you however don't remember the password, but being the mathematician you are(by creed of the bounty hunters guild) you had noted down the question whose answer is the password:
For each positive integer k, let $P_k(x)$ be the unique polynomial such that $P_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$ for all $n \in \mathbb{N}$.
For example, $P_1(x) = \dfrac{x(x+1)}{2}$ is the unique polynomial for which
$P_1(n) = 1 + 2 + \cdots + n$ for all naturals n.

The password is the value of $\left(1 + \left\lceil P_{2022}\left(\dfrac{2023}{2022}\right)\right\rceil\right)^{\left(P_{2024}\left(\frac{-1}{2}\right)+P_{2023}(-1)\right)}$.
(300 Points)

Problem 3

The door opens, revealing the bustling city of MOS ESPA some distance away from you. you make your way into the city and head into the nearest bar. While you order, a villager makes his way to you and says, "I am the magistrate of this city, you ,a bounty hunter, is exactly what the city of MOS ESPA needs in these trying times." you nod him to the seat in front of you while he goes on. "you see, for the past month or so desert pirates have been raiding and looting our city relentlessly and we would love it if you would help us eliminating them. I would pay a large sum for this amount." Before you accept the quest, he interjects, "But if I am to be sure to trust you, I must test you. I know your creed prides itself on its mathematical prowess, so I pose you this question. Prove to me that the following equation has no integer solutions" **(300 Points)**

$$m^5 + 3m^4n - 5m^3n^2 - 15m^2n^3 + 4mn^4 + 12n^5 = 33$$

Problem 4

Having accepted the magistrate's quest you make your way to a workshop to repair your ship and stock up on resources and weapons. Nearing the workshop you hear disgruntled screams and wander in. you see a woman bent over a service droid banging it with a wrench. Noticing you, she calls out, "Hey! Instead of standing there like a dim wit, why don't you come over here and help me out?" Startled you go towards her. "I can repair anything from star ships to cruisers" she says, "but goddamn droids is something I could never do. Maybe you could help me out? I need to find out the number of binary sequences $(a_0, a_1 \ldots a_n)$ such that:"

$$a_0 \leq a_1 \geq a_2 \leq a_3 ...$$

.

Help the woman in fixing her droid. **(200 Points)**

Problem 5

Having helped the woman, she turns to you gratefully. you tell her that you need your ship to be repaired. To bring back the ship. She has multiple helper droids with different lifting capacities but she can only bring three droids to help her. Let the lifting capacities be x, y and z such that $x \geq y \geq z \geq 0$. Note that x, y, z are integers. They follow the following relation:

$$x^2 + y^2 + z^2 = 2(xy + yz + zx)$$

$$x^2 + \sqrt{3660 - y - z - 2(\sqrt{yz})} = 3660$$

Find all possible triples (x, y, z) satisfying the above equation that she can bring with her. **(200 Points)**

Problem 6

When you reach the ship, the ship appears to be striped apart. "Ah 'tis the work of the Jawas, those little rodent scavengers, they seemed to have stripped all your ships parts" the woman kneels down to look at the damage done by the Jawas. "You'll need to get it back from them" she says. "Search for the Sandcrawler, the home/ship of the Jawas. However, searching for the Sandcrawler will take you ages, why dont you ponder over this question ive been struggling with?" she asks you:

Let a,b be natural numbers and let p be a prime. Find all possible triples (a,b,p) such that the following holds **(500 Points)**

$$p = \frac{b}{4}\sqrt{\frac{2a - b}{2a + b}}$$

Problem 7

Zooming across the desert over the setting twin suns on a speeder bike you catch sight of the Sandcrawler trudging across the sands. you stop in front of the entrance and a panel lights up with some words written on it in a language foreign to you. Gazing at it and scanning it with your helmet you notice that it is a math problem:

"Let A denote the collection of real numbers of the form $a+b\sqrt{93}$ where $a, b \in \mathbb{Q}$. Let B denote the collection of real numbers which are roots of monic (leading

coefficient 1) polynomials with integer coefficients. Find $A \cap B$."

you realize that to gain entrance into the might ship, you need to solve the problem. **(300 Points)**

Problem 8

Having allowed entrance into the the sandcrawler, you are surrounded by the 3 feet tall Jawas. you, unfortunately realize that you dont understand Jawaese. As a last ditch effort, you decide to try to teach them the meaning of some mathematical symbols so that you can still talk about some problems which might arise while fixing your ship.After hours of effort you have communicated the meaning of the following symbols to the Jawas $=,\rightarrow,\wedge,\vee,\neg,\forall,\exists$.. you have also made them understand that in our language we use letters to represent variables and how we use parenthesis. Next to start off you try to discuss some properties of the natural numbers.

a) Can you write a sentence that says there are at least 2 natural numbers?**(25 Points)**

b) you notice that a Jawan only uses the following symbols while talking with you $=,\wedge,\neg,\forall$.Does there always exist a sentence written with these set of symbols with the same meaning as any given sentence with the symbols you translated? **(50 Points)**

c) you manage to make them understand the meaning of $\leq$. Can you now write a sentence that is true only for $x = 0, 1, 2$?**(25 Points)**

d) The symbols, $+,\cdot,0,1$ have also been translated. you notice that the Jawas are more knowledgeable than they appear. Naturally the mathematician in you is curious to know if they have solved the twin prime conjecture - the assertion that their are infinitely many pairs of primes that differ by 2. Can you write a sentence stating the twin prime conjecture?**(100 Points)**

NOTE: The set that $\forall$ and $\exists$ are quantifying from is understood from the context, for example here if we wanted to communicate that for every natural number y, y is equal to y we would write $\forall y\ (y = y)$

Now finally you begun discussing some properties of the real numbers. But the earlier incident in part b) makes you wonder if it's possible to write some of your sentences with fewer symbols.

e) Is it possible to write a sentence equivalent to $\exists x\ ax \cdot x + bx + c = 0$ without using the symbols $\forall$ and $\exists$?**(25 Points)**

f) Is it possible to write a sentence equivalent to $\exists x \exists y\ (ax + by = 0) \wedge (cx + dy = 0) \wedge \neg(x = 0)$ without using the symbols $\forall$ and $\exists$? **(25 Points)**

NOTE: a,b,c,d are some constants that have been translated