Problem 1

1. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. Let $G = \{(x, f(x)) : x \in \mathbb{R}\}$. Then which of the following statements are true?

(a) There exist $c \in \mathbb{R}$ and an infinite subset $S \subseteq \mathbb{R}$ such that $f(x) = cx \ \forall \ x \in S$.

(b) If f is monotonic on $\mathbb{R}$, then there exists $c \in \mathbb{R}$ such that $f(x) = cx \ \forall \ x \in \mathbb{R}$.

(c) If there exists a point in $\mathbb{R}$ at which f is not continuous, then G is dense in $\mathbb{R}^2$, that is, for every $(a, b) \in \mathbb{R}^2$ and every $r > 0$, the set $G \cap \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < r^2\}$ is not empty.

(d) The set $\{x \in \mathbb{Q} : f(x) \notin \mathbb{Q}\}$ is either empty or infinite.

Problem 2

2. How many pairs (x, y) of non-negative integers satisfy the equation $(xy-7)^2 = x^2 + y^2$?

(a) 2 (b) 0 (c) 4 (d) 5

Problem 3

3. Let $\mathbb{R}_0$ be the set of non-negative real numbers. Let S be the set of all functions $f : \mathbb{R}_0 \to \mathbb{R}_0$ that satisfy $f(f(x) - x) = 2x$ for all non-negative real numbers x. Then which of the following statements are true?

(a) S is infinite.

(b) $|S| = 1$

(c) All functions in S are continuous at every point in $\mathbb{R}_0$.

(d) For all functions f in S, the function $g : \mathbb{R}^+ \to \mathbb{R}_0$ defined by $g(x) = \dfrac{f(x)}{x}$ is bounded.

Problem 4

4. An integer a is said to be a *multiplicative inverse modulo* n of an integer b if $ab \equiv 1$ $(mod\ n)$. Let $f(n)$ be the number of integers in $\{1, .., n\}$ that have

a multiplicative inverse modulo n. Then which of the following statements are true?

(a) There exists an integer N such that $f(n)$ is even for all $n \geq N$.

(b) $91^{f(66)} \equiv 5 \ (mod\ 66)$

(c) The set $\{n : \sum_{d|n} f(d) > n\}$ is infinite.

(d) $51^{f(140)} \equiv 1 \ (mod\ 140)$

Problem 5

5. In a lottery, tickets are given nine-digit numbers using only the digits 1,2,3. They are also coloured red, blue or green in such a way that two tickets whose numbers differ in all the nine places get different colours. Suppose the ticket bearing the number 122222222 is red and that bearing the number 222222222 is green. What is the colour of the ticket bearing the number 331331331?

(a) Red (b) Blue (c) Green (d) Insufficient Information

Problem 6

6. Consider the following functional equation:
$f : \mathbb{R} \to \mathbb{R},\ f(x+y)f(x-y) = (f(x) + f(y))^2 - 4x^2 f(y)\ \forall x, y \in \mathbb{R}$. Which of the following statements are true?

(a) Infinitely many functions satisfy the equation.

(b) Exactly two functions satisfy the equation.

(c) There exists a discontinuous function that satisfies the equation.

(d) There exists an even function that satisfies the equation.

Problem 7

7. Suppose p and q are primes and $n > 2$ is an even number such that $p^n + p^{n-1} + ... + p + 1 = q^2 + q + 1$. Then which of the following are pos-

sible values of $(p+q+n)$?

(a) 33 (b) 11 (c)27 (d) 13

Problem 8

Let there be 2 functions f,g such that

$$f^{(i)}(0) = 2^i g^{(i)}(0) \quad 1 \leq i \leq n$$

$$f^{(n)}(x) < 2^{nx} \text{ and } g^{(n)}(x) < 2^x$$

The for any y in the image of both f and g, what is the value of $\frac{f^{-1}(y)}{g^{-1}(y)}$

Problem 9

Bob is climbing a slippery staircase and at each step he makes a move. A move either leads him to the next step or he slips to the previous step (both being equally likely). Bob falls to death if he slips off the first step at any stage. Bob is initially at the first step of the staircase. Alice is waiting for him at the 2023^{rd} step. What is the probability that he will meet Alice? (Of course, once he reaches the 2023^{rd} step, he stops.) **Note:** Since it is probability, write the answer in fractions (for eg. 1/2)

Problem 10

Let $S = \{(f,g) \in \mathscr{F}^2 : f(g(x)) = x^{2021} \text{ and } g(f(x)) = x^{2022}\}$ where $\mathscr{F}$ is the set of all functions $h : \mathbb{R} \to \mathbb{R}$. Find $|S|$.